relations among the R which are identities in the determinants. We proceed to the classification.

There exists no relation which is not either itself homogeneous in the elements of a certain set of rows, and of a certain set of columns, or obtainable by the composition of relations having these properties. The most general relation can be regarded as linear in the determinants of some set and as having for coefficients rational integral functions of the elements. Let $\Sigma C \triangle \equiv 0$ be such a relation. Choose a determinant containing a set of columns $(i, j, k, \ldots\ldots)$. Multiply these columns in the array by $s_i, s_j, s_k, \ldots\ldots$ The result must vanish for every s. Hence equating to zero the coefficients of the various powers and products of the s's we obtain a set of identities

$$\Sigma C_c \triangle_c = 0$$

homogeneous in a certain set of columns. A similar process in the rows gives a set of identities $\Sigma C_{cr} \triangle_{cr} = 0$, when each term contains the same set of rows and columns. Without loss of generality we may set $m \geqq n$, where there are m rows and n columns. The original identity depends on and is composed of these. Since a single determinant cannot vanish identically the coefficient of each determinant must contain an element of some row not represented in this determinant but in others.

First consider possible relations between the $n+1$ determinants which enter a single set of $n+1$ rows. These can be obtained from a vanishing determinant whose jth row is

$$A_{j_1} a_{j_1} + A_{j_2} a_{j_2} + \ldots\ldots A_{j_n} a_{j_n}, \quad a_{j_1}, a_{j_2}, a_{j_3}, \ldots\ldots, a_{j_n}.$$

Consider the special case $a_{1k}=1$, $a_{2k}=a^k$, $a_{3k}=b^k$, ..., $a_{n+1,k}=z^k$, and put first $a=1$. We have

$$(A_{11}+A_{12}+\ldots\ldots)[(1, b, c^2, \ldots\ldots, z^n)] = (A_{21}+A_{22}+\ldots\ldots)[(1, b, c^2, \ldots\ldots, z^n)]$$

or $\Sigma A_{1k} = \Sigma A_{2k}$, for $(1, b, c^2, \ldots\ldots, z^n) \neq 0$. So in general $\Sigma A_{jk} = \Sigma A_{1k}$.

Returning to the general form, the A_{jk} each lack a kth column of being homogeneous in columns and a jth row of being homogeneous in rows. Hence, if in $\Sigma A_{1k} = \Sigma A_{2k}$ we place all the elements of the 1st column $=0$ we obtain $A_{11} = A_{21}$, and similarly $A_{jk} = A_{1k}$, and the relation reduces to a series of multiples of the lineo-linear relations.

Secondly we treat the case where determinants of more than $n+1$ rows enter.

Lemma. Out of m different things r sets of n things are taken, and for each of these sets of n, every set of n+1 including it is formed. If the number of these sets is s and $m-n \geq 2$ and $m \geq 2$, then $ns > r$.

Let the r sets be $a_{j_1}, a_{j_2}, a_{j_3}, \ldots\ldots, a_{j_n}$ $(j=1, 2, \ldots\ldots, n)$. Form the sets

$$(a_{j_1}, a_{j_2}, \ldots\ldots, a_{j_n})a_t \quad (j=1, 2, \ldots\ldots, r;\ t=1, 2, \ldots\ldots, n).$$

Of these nr have repeated elements leaving $(m-n)r$ which may in the complete system occur any number of times up to $n+1$. If T_f be the number occurring f times, $s=\Sigma \frac{T_f}{f}$ $(f=1, \ldots, n+1)$; $\Sigma T_f=(m-n)r$. Therefore

$$ns=\Sigma\frac{nT_f}{f}>\frac{n(m-n)r}{n+1}>r.$$

Now let determinants of more than one set of $n+1$ rows enter the relation. If s be the number of these sets from the ns lineo-linear relations belonging to the sets, multiply them by indeterminate coefficients, add and equate the coefficient of each determinant to the corresponding coefficient in the given relation. There are r linear homogeneous equations between ns indeterminates. They are independent; for, each involves a given set of rows, and as $ns>r$, rational integral functions of the elements can be found (in general in more than one way) for the indeterminates. This reduces the given relation to a sum of multiples of the lineo-linear relations.

We consider primarily then "vanishing aggregates of determinant products" of the second order. Those of the third order vanish only by being sums of products of determinants by such second order aggregates. The third order aggregates of which Monge* gave the first example, are important as relations used in elucidating the plexus between those of those of the second order.

A vanishing aggregate of determinant products of the second order, is then homogeneous in rows and columns, of order $2n$ in the elements. It contains at least three terms. For, the product of the determinants containing the rows $p, q, r, \ldots, w$ and $s, t, u, \ldots, z$, respectively, has a term

$$a_p\, b_q\, c_r, \ldots, l_w \,.\, a_s\, b_t\, c_u, \ldots, l_z$$

which can only be cancelled by a product from determinants such as $(p, t, r, \ldots, w)(s, q, u, \ldots, z)$ formed by simple interchange from first product, and this in turn has a term $a_t\, b_p\, c_r, \ldots, l_w \,.\, a_q\, b_s\, c_u, \ldots, l_z$ which does not occur in the first product.

We take a set of R such relations $(R \geq I)$ given by systematic enumeration and such that in every case these R are necessary and sufficient relations, but in any particular case (*i. e.* some one determinant not zero) a definite I of these relations are the independent ones. The set R may then be considered as fundamental.

First we have relations of $n+1$ terms given by the expansion of the identically vanishing determinant

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*Muir, p. 68.

$$\begin{vmatrix}
a_1 & b_1 & c_1 \ldots\ldots l_1, & 0 & 0 & 0 \ldots\ldots 0 \\
a_2 & b_2 & c_2 \ldots\ldots l_2, & 0 & 0 & 0 \ldots\ldots 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_{n-1} & b_{n-1} & c_{n-1} \ldots\ldots l_{n-1}, & 0 & 0 & 0 \ldots\ldots 0 \\
a_n & b_n & c_n \ldots\ldots l_n, & a_n & b_n & c_n \ldots\ldots l_n \\
a_{n+1} & b_{n+1} & c_{n+1} \ldots\ldots l_{n+1}, & a_{n+1} & b_{n+1} & c_{n+1} \ldots\ldots l_{n+1} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_{2n} & b_{2n} & c_{2n} \ldots\ldots l_{2n}, & a_{2n} & b_{2n} & c_{2n} \ldots\ldots l_{2n}
\end{vmatrix}$$

As the choice of rows and of rows set opposite zeros is alone material we may denote this by

$$\frac{0,\ 0,\ 0, \ldots, 0,\ n \mid n+1,\ n+2, \ldots, 2n}{1,\ 2,\ 3, \ldots, n-1,\ n \mid n+1,\ n+2, \ldots, 2n}$$

There are $\binom{m}{2n}\binom{2n}{n-1}$ of these, except for $n=2$ where the relation remains unchanged whichever row is placed opposite the zero. For $n=2$ the number is $\binom{m}{4}$.

This class of relations differs from the other classes to be discussed in important points. Since all the rows are different each determinant is multiplied by the same complementary determinant in each of the $\binom{2n}{n-1}$ relations formed from a given $2n$ rows. We may refer to this class as the *complementary class.*

Next we have relations of r terms $2<r<n+1$, given by the expansion of a similar vanishing determinant whose symbol is

$$\frac{0,\ 0, \ldots, 0,\ n \mid n+1,\ n+2, \ldots, n+r-1,\ 1,\ 2, \ldots, n-r+1}{1,\ 2, \ldots, n-1,\ n \mid n+1,\ n+2, \ldots, n+r-1,\ 1,\ 2, \ldots, n-r+1}$$

The number is

$$\binom{n}{n+r-1}\binom{n+r-1}{n-r+1}\binom{2r-2}{r-2},$$

except for $r=3$ when the last factor is replaced by 1. The total for $r=3, 4, \ldots, n+1$ is

$$R = \binom{m}{n-1}\left[\binom{m}{n+1} - \frac{(m-n+1)(m-n)(mn-n^2-m+5)}{8}\right].$$

For $r<n+1$ the relations are "extensionals" of the relations of the complementary class for $m'=2n'$, where m' is the number of rows not used as "extenders." For example, $\frac{0\ 0\ 0\ 4 \mid 5\ 6\ 7\ 1}{1\ 2\ 3\ 4 \mid 5\ 6\ 7\ 1}$ may be regarded as derived from $\frac{0\ 0\ 4 \mid 5\ 6\ 7}{2\ 3\ 4 \mid 5\ 6\ 7}$ by using 1 as an "extender."

If there is an identical relation between complementary relations involving the same set of rows of the form $R_1+R_2+R_3=\ldots\ldots=0$, this relation may be extended by introducing any set of new rows into all the determinants entering. For example, the relation of complementary type of the 3×6 array are connected (among other connections) by

$$\frac{0\ 0\ 1\ |\ 4\ 5\ 6}{1\ 2\ 3\ |\ 4\ 5\ 6}+\frac{0\ 0\ 1\ |\ 2\ 5\ 6}{3\ 4\ 1\ |\ 2\ 5\ 6}+\frac{0\ 0\ 1\ |\ 2\ 3\ 4}{5\ 6\ 1\ |\ 2\ 3\ 4}\equiv 0,$$

which vanishes in the determinants. The relation

$$\frac{0\ 0\ 0\ldots\ldots 0\ 0\ 3\ |\ 4\ 5\ 6\ a\ b\ c\ \ldots\ldots}{a\ b\ c\ldots\ldots 1\ 2\ 3\ |\ 4\ 5\ 6\ a\ b\ c\ \ldots\ldots}+\frac{0\ 0\ 0\ldots\ldots 0\ 0\ 1\ |\ 2\ 5\ 6\ a\ b\ c\ \ldots\ldots}{a\ b\ c\ldots\ldots 3\ 4\ 1\ |\ 2\ 5\ 6\ a\ b\ c\ \ldots\ldots}$$

$$+\frac{0\ 0\ 0\ldots\ldots 0\ 0\ 1\ |\ 2\ 3\ 4\ a\ b\ c\ \ldots\ldots}{a\ b\ c\ldots\ldots 5\ 6\ 1\ |\ 2\ 3\ 4\ a\ b\ c\ \ldots\ldots}\equiv 0$$

also holds, when $a\ b\ c\ \ldots\ldots$ is any set of rows. For, the vanishing of the simple relation depends on interchanges of rows, which interchanges are not affected by the extension.

Given any relation of complementary type

$$\frac{0\ 0\ 0\ldots\ldots(r-1)\ \ |\ \ r,\ r+1,\ \ldots\ldots,\ 2(r-1)}{1\ 2\ 3\ldots\ldots(r-1)\ \ |\ \ r,\ r+1,\ \ldots\ldots,\ 2(r-1)},$$

involving r terms, we obtain by extension $\binom{m-2r+2}{n-r+1}$ new relations. There are $\binom{m}{2(r-1)}\binom{2(r-1)}{r-2}$ such complementary relations, so that

$$\binom{n-r+1}{m-2r+2}\binom{m}{2(r-1)}\binom{2(r-1)}{r-2}=\binom{m}{n+r-1}\binom{n+r-1}{n-r+1}\binom{2r-2}{r-2}$$

is the number of relations of r terms in general.

The number of relations involving a given determinant is

$$K=\sum_{r=4}^{r=n+1}\left[2(r-1)\binom{m-n}{r-1}\binom{n}{r-1}\right]+\binom{m-n}{2}\binom{n}{2}.$$

For three-termed relations, we have the $(n-2)$nd extensions of the single relation $\frac{0\ b\ |\ c\ d}{a\ b\ |\ c\ d}$. $\binom{m-n}{2}$ is a factor of the number for the two rows in the given determinant, and $\binom{n}{2}$ a factor for choice of two rows not in the extenders. For $r>3$ we have a factor for the order of arrangement of $\frac{0\ 0\ldots\ldots c\ |\ d\ e\ldots\ldots f}{a\ b\ldots\ldots c\ |\ d\ e\ldots\ldots f}$. If

$a, b, \ldots, c$ belong to the given determinant, $(r-2)$ or none of these rows occur opposite the zeros. This gives $2(r-1)$ as the factor, and K as the total.

The relations of more than three terms involving a given determinant k are not independent but are connected with one another and those of their own type not involving k by linear relations whose coefficients are ± 1.

Take first the case $m=2n$ and the relations of $n+1$ terms. Write the $2n$ column numbers as

$$1, 2, 3, \ldots, n, \bar{n}, (\overline{n-1}), \ldots, \bar{3}, \bar{2}, \bar{1}.$$

The relation

$$\begin{array}{l} 0, 0, 0, \ldots\ldots, 0, n \mid \bar{n}, (\overline{n-1}), \ldots, \bar{2}, \bar{1} \\ 1, 2, 3, \ldots, (n-1), n \mid \bar{n}, (\overline{n-1}), \ldots, \bar{2}, \bar{1} \end{array}$$

contains $k=(1, 2, \ldots, n)$. So does each relation arising out of this by any interchanges of $1, 2, \ldots, n$; if $\bar{n}, (\overline{n-1}), \ldots, \bar{2}, \bar{1}$ are interchanged also, r and $\bar{r}$ going together, the leading terms have all the same sign. In general these interchanges do not alter the signs. The corresponding relation

$$\begin{array}{l} 0, \quad 0, \ldots\ldots, 0, 1 \mid 2, 3, \ldots, n, \bar{n} \\ (\overline{n-1}), (\overline{n-2}), \ldots, \bar{1}, 1 \mid 2, 3, \ldots, n. \bar{n} \end{array}$$

and all the results of interchanges also contain k. If $n>2$ none of this set falls in the last mentioned. Add all these relations; the result is not an identity in the relations but vanishes in the determinants. For example, the products $[1, 2, \ldots, (n-1), \bar{a}][n, \bar{n}, \ldots, (\overline{a+1}), (\overline{a-1}), \ldots, \bar{1}]$ and $[(\overline{n-1}), \ldots, (\overline{a+1}), \bar{a}, (\overline{a-1}), \ldots, \bar{1}, 1][2, 3, \ldots, n, \bar{n}]$ occur in the two sets. By the interchange $(1n)$ followed by $(1a)$, the latter becomes

$$[(\overline{n-1}), \ldots, (\overline{a+1}), \bar{1}, (\overline{a-1}), \ldots, \bar{2}, \bar{n}]$$
$$[2, 3, \ldots, (a-1), 1, (a+1), \ldots, (n-1), a, \bar{a}]$$

which is identical, except as to sign, with the first product. By suitable interchanges this pair of terms becomes other pairs and exhausts the two sets together. Hence we have an identity which may be written $\Sigma R(k)=0$ $(m=2n)$. A single interchange of a barred and unbarred column changes all the $R(k)$'s but two to $R(\bar{k})$'s, the exception being the two free from the interchanged columns. This interchange can be chosen so that for any one of the first set we have any one of the second left unchanged, and by subtraction we have for every pair of $R(k)$'s:

$$R_1(k)+R_2(k)+\Sigma R(\bar{k})=0 \ (m=2n).$$

This relation can now be extended by any set of columns. As no such relations hold for the 2×4 array the extended relations hold only down to R_4.

The reduction of the $R(\bar{k})$ to dependence on the $R(k)$ can be accomplished as follows. Let

$$R(\bar{k}) = \frac{0\ 0\ 0\ 0 \ldots\ldots z \mid \alpha\ \beta\ \gamma \ldots\ldots \varsigma}{a\ b\ c\ d \ldots\ldots z \mid \alpha\ \beta\ \gamma \ldots\ldots \varsigma}, \quad k=(1, 2, \ldots, n).$$

Form the $3n$ row determinant whose upper right $2n \times 2n$ places are occupied by $R(\bar{k})$ and lower left $n \times n$ by k and fill in zeros below the zeros of $R(\bar{k})$ letters below the letters. Repeat k twice above itself. The symbol of it may be taken as

$$\begin{array}{l|l|l} 1\ 2 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 0\ z & \alpha\ \beta\ \gamma\ \delta \ldots\ldots \varsigma \\ 1\ 2 \ldots\ldots n & a\ b\ c \ldots\ldots y\ z & \alpha\ \beta\ \gamma\ \delta \ldots\ldots \varsigma \\ 1\ 2 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 0\ z & \alpha\ \beta\ \gamma\ \delta \ldots\ldots \varsigma \end{array}$$

Expanding by lower n rows, we get $D = k\,R(\bar{k}) + \Sigma\, a\, R$, when every R has at least one column which occurs in k. This determinant vanishes identically in the elements but not necessarily in the determinants. Expanding by the middle n rows, we see that only the determinants $[a\,b\,c \ldots\ldots y\,X]$ have non-vanishing coefficients. If X comes from the columns to the right the coefficient is 0 for n odd, and $2\,R(k)$ for n even. If X comes from k the coefficient contains $(n-1)$ columns of k if it does not vanish, and is expressible as $\Sigma\, R'_{n-1}$. We may then write

$$k\,R(\bar{k}) \equiv \Sigma\, a\, R_1 + \Sigma\, b\, R_{n-1} + \Sigma\, c\, R(k)$$

when every R_1 contains one column of k not opposite a zero. R'_{n-1} has $(n-1)$ columns of k in some position. The result is an identity in the determinants.

Take one of the R_1 and form a new $3n$ rowed determinant

$$\begin{array}{l|l|l} 1\ 2\ 3 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 1 & \alpha\ \beta\ \gamma \ldots\ldots \varsigma \\ 1\ 2\ 3 \ldots\ldots n & a\ b\ c \ldots\ldots y\ 1 & \alpha\ \beta\ \gamma \ldots\ldots \varsigma \\ 1\ 2\ 3 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 0\ 0 & \alpha\ \beta\ \gamma \ldots\ldots \varsigma \end{array}$$

Expanding by middle rows as before, we get $D \equiv \Sigma\, a\, R(k) + \Sigma\, b\, R'_n$. By lower rows, we reach

$$k\,R(\bar{k}) \equiv \Sigma\, a\, R_2 + \Sigma\, b\, R'_n + \Sigma\, c\, R(k).$$

Finally we reach R_{n-1}, and, by working on R'_{n-1} and on R'_n, we reduce them to R_{n-1}, *i. e.*, to relations involving $n-1$ rows of k not opposite the zero.

We then apply

$$\begin{array}{l|l|l} 1\ 2 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 0\ 1 & 2\ 3 \ldots\ldots (n-1)\ \varepsilon\ \varsigma \\ 1\ 2 \ldots\ldots n & a\ b\ c \ldots\ldots y\ 1 & 2\ 3 \ldots\ldots (n-1)\ \varepsilon\ \varsigma \\ 1\ 2 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 0\ 0 & 0\ 0 \ldots\ldots 0\quad \varepsilon\ \varsigma \end{array}$$

The $3n$ rowed determinant vanishes in the elements; for, subtraction of the upper n rows from middle leaves only $(n-1)$ columns. This time $D = \Sigma\, a\, R(k) + \Sigma\, b\, R_{n-1}$, when the symbols giving rise to R_{n-1} contain only two columns foreign to k and either vanishes identically or contains k. Expanding by lower rows, we get $D \equiv k\,R(\bar{k}) - \Sigma\, a\, R(k)$. Hence each $R(\bar{k})$ is reduced by a chain to depend on $R(k)$.

Now assume that all the $R_3(k)$ and one $R_r(k)$ in each set of columns vanish. The number of relations assumed satisfied is I, the total number of independent relations. Apply a series of reducing operations to

$$R_r(k)+R'_r(k)=\Sigma R''_r(k).$$

The $R''_r(k)$ have already n columns of k, and one more is introduced not opposite the zero. The result is an extensional with one more column among the extenders, *i. e.*, is an $R''_{r-1}(\bar{k})$. Hence the $R'_r(k)$ do not reappear in the chain. In treating the $R(\bar{k})$ it may happen that an extender is replaced on one side and and $R_r(\bar{k})$ lead to $R_{r+1}(\bar{k})$ or even to an $R_{r+1}(k)$, but these are reducible again, and eventually reduce to $R_3(k)$.

We finally have the theorem: *If* $k\neq 0$, *every relation is satisfied if every* $R_3(k)=0$, *and one* $R_r(k)=0$ *for every r and every set of columns.*

The number of independent determinants ot order n in an $m\times n$ array is thus in general $D=n(m-n)+1$.

In the particular case of zero values, however, either all vanish or there is an $(n-1)\times n$ array in which not all the minors of order $(n-1)$ vanish. Hence Cayley's theorem holds: $D_0=m-n+1$. If the $m-n+1$ determinants with $n-1$ columns of this array in common vanish, all others vanish. A general proof follows. A determinant not containing columns 1, 2, ..., $n-1$ has at least two columns not in this set. Hence the relation

$$\frac{0\ 0\ \ldots\ldots\ 0\ \alpha\ |\ \beta\ \gamma\ \ldots\ldots\ \sigma\ M}{1\ 2\ \ldots\ n-1\ \alpha\ |\ \beta\ \gamma\ \ldots\ldots\ \sigma\ M}=0,$$

gives $(\alpha\ \beta\ \gamma\ \ldots\ldots\sigma)(1\ 2\ 3\ (n-1)M)=\Sigma(1\ 2\ 3\ \ldots\ldots n-1\ \alpha)(M\ \beta\ \gamma\ \ldots\ldots\sigma)$ if M be an arbitrary column and every minor of order $n-1$ from $(1, 2, \ldots, n-1)$ be not zero, $(1, 2, \ldots, n-1, M)\neq 0$. Then from $(1, 2, 3, \ldots, n-1, \alpha)=0$, α any of $m-n+1$ columns, follows $(\alpha\ \beta\ \gamma\ \ldots\ldots\sigma)=0$. As in the general case $m-n\geq 2$.

There are also relations between determinants of different orders taken from an array. The vanishing determinant

$$\left|\begin{array}{ccccccc}
a_1 & b_1 \ldots\ldots f_1, & a_1 & b_1 & c_1 \ldots\ldots & l_1 \\
a_2 & b_2 \ldots\ldots f_2, & a_2 & b_2 & c_1 \ldots\ldots & l_2 \\
\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
a_{n+1} & b_{n+1}\ f_{n+1}, & a_{n+1} & b_{n+1} & c_{n+1}\ldots\ldots & l_{n+1} \\
0 & 0 \ldots\ldots 0 & a_i & b_i & c_i \ldots\ldots & l_i \\
0 & 0 \ldots\ldots 0 & a_j & b_j & c_j \ldots\ldots & l_j \\
\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
0 & 0 \ldots\ldots 0 & a_k & b_k & c_k \ldots\ldots & l_k
\end{array}\right|$$

($r-1$ rows; r columns)

gives r types of identities between $(n\times n)$ and $(r\times r)$ determinants. If s of the

set $(i, j, \ldots, n+1)$, there are $\begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} m \\ n+1 \end{bmatrix}\begin{bmatrix} n+1 \\ s \end{bmatrix}\begin{bmatrix} m-n-1 \\ r-s-1 \end{bmatrix}$ of the type s $s=[0, 1, \ldots, (r-1)]$. If the array have M rows and N columns the total number of type s is $\begin{bmatrix} M \\ N \end{bmatrix}\begin{bmatrix} N \\ r \end{bmatrix}\begin{bmatrix} M \\ n+1 \end{bmatrix}\begin{bmatrix} n+1 \\ s \end{bmatrix}\begin{bmatrix} M-n-1 \\ r-s-1 \end{bmatrix}$. These relations between $(n \times n)$ and $(r \times r)$ determinants, $n > r$, can be reduced to sums of products of determinants of $(n-r)^2$ elements and the relations between determinants of the same order r added to products of $(n-r)$th order determinants and identities between determinants of order r which vanish in the determinants.

SEVERAL FUNDAMENTAL THEOREMS IN GROUP THEORY.

By DR. G. A. MILLER.

The following theorems are readily deduced from known theorems. They appear to be of sufficient importance to be explicitly stated in view of their numerous elementary applications.

Theorem I. *If a group contains more than one cyclic subgroup of order* $k=p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots\ldots, p_\lambda^{\alpha_\lambda}$, *where* $p_1, p_2, \ldots\ldots, p_\lambda$ *are distinct prime numbers such that* $p_1 < p_2 < \ldots\ldots < p_\lambda$, *then it contains at least* p_1 *such subgroups. If the number of these subgroups is exactly* p_1 *the value of* α_1 *exceeds unity.*

This theorem may readily be derived from the known theorems that the number of cyclic subgroups of order p^α, $\alpha > 1$ and $p > 2$, in a non-cyclic group of order p^m is always divisible by p, and that the total number of subgroups of order p^α in such a group is of the form $1 + mp$. Suppose that a group G contains more than one cyclic subgroup of order k but that the number of these subgroups is less than p_1+1. Each of these subgroups is transformed into itself by every other one, since an operator of order p^β transforms things in multiples of p when it does not transform them into themselves.

From this it follows that two of these cyclic subgroups generate a group which is the direct product of its Sylow subgroup. Since these Sylow subgroups cannot all be cyclic it is proved that this direct product contains at least p_1 cyclic subgroups of order k, and hence G contains at least p_1 cyclic subgroups of this order. From the fact that a group of order p^m cannot contain exactly p cyclic subgroups of order p^α when $\alpha = 1$, it follows directly that $\alpha_1 > 1$ whenever G contains exactly p_1 cyclic subgroups of order k.

As an interesting particular consequence of this theorem we have that there is no group which contains exactly two cyclic subgroups of the same order when this order is either odd or twice an odd number. Moreover, it is easy to construct groups containing exactly two cyclic subgroups of any arbitrary order which is divisible by 4. For instance, there is an infinite number of groups containing just two cyclic subgroups of order twelve, but there is not a single

group which contains just two cyclic subgroups of order six. Similar remarks evidently apply to the other prime numbers.

Theorem II. *If a group contains exactly p cyclic subgroups of order k, then it contains only one subgroup of order* p^{a_λ}, $\lambda > 1$, *and hence its Sylow subgroups whose orders are divisible by* p_λ *are cyclic.*

If these Sylow subgroups were not cyclic G would contain at least p_λ subgroups of order $p_\lambda^{a_\lambda}$. This is impossible since each of these subgroups would transform into itself each of the p_1 cyclic subgroups of order k, and hence it would also transform each of the operators of such a subgroup, with a possible exception of those whose orders are divisible by p_λ, into itself. From this it follows directly that there would be more than p_1 cyclic subgroups of order k in G. In the same manner it may be observed that if the order of G is divisible by any prime which exceeds p_λ all its operators whose orders are powers of this prime are commutative with each operator in the p_1 cyclic subgroups of order k.

Theorem III. *A necessary and sufficient condition that a group is the direct product of its Sylow subgroups is that the nth power of each of its operators is contained in every subgroup of index n, for every possible value of n.*

That this condition is sufficient follows directly from the fact that if G would contain $1+kp$ Sylow subgroups of order p^m all the operators which would transform such a subgroup into itself would constitute a subgroup of index $1+kp$. As the latter would not contain all the operators whose orders are powers of p it could not contain the $1+kp$ power of every operator of G.

That the condition is also necessary follows from the fact that every subgroup of order p^a which is contained in a group of order p^m is itself invariant in a subgroup of order p^{a+1} and hence it includes the pth power of all the operators of the latter subgroup. Similarly, the latter subgroup contains the pth power of all the operators of a subgroup of order p^{a+2} and hence the given subgroup of order p^a contains the p^2 power of all the operators in this subgroup of order p^{a+2}. This reasoning can clearly be continued until we arrive at the entire group of order p^m.

It is well known that another necessary and sufficient condition that G is the direct product of its Sylow subgroups is that we arrive at the identity by finding the group of cogredient isomorphisms of G, and then finding the group of cogredient isomorphisms of this group of cogredient isomorphisms, and then the third group of cogredient isomorphisms, etc. A group which is its own group of cogredient isomorphisms contains no invariant operator besides the identity, and vice versa.

DEPARTMENTS.

DISCUSSION.

THE EVALUATION OF $\int_0^\pi \frac{\sin mx}{x}dx$.

By S. A. COREY, Hiteman, Iowa.

The following solution of problem 203, Calculus, the evaluation of the definite integral $\int_0^\pi \frac{\sin mx}{x}dx$ (m an integer), involves several points of interest.

$$\int_0^\pi \frac{\sin mx}{x}dx = \int_0^{m\pi}\frac{\sin x}{x}dx.$$

Developing $\int \frac{\sin x}{x}dx$ by the formula,*

$$f(x)=f(0)+\frac{x}{r2}\left\{[f'(x)+f'(0)]+2\left[f'\left[\frac{x}{r}\right]+f'\left[\frac{2x}{r}\right]+f'\left[\frac{3x}{r}\right]+\cdots\cdots\right.\right.$$

$$\left.\left.+f'\left[\frac{r-1}{r}x\right]\right]\right\}-\frac{B_1 x^2}{r^2.2!}[f'(x)-f'(0)]+\frac{B_2 x^4}{r^4.4!}[f^{iv}(x)-f^{iv}(0)]$$

$$-\frac{B_3 x^6}{r^6.6!}[f^{vi}(x)-f^{vi}(0)]+\cdots\cdots+(-1)^n\frac{B_n x^{(2n)}}{r^{(2n)}.(2n)!}[f^{(2n)}(x)-f^{(2n)}(0)]+\cdots\cdots(1),$$

(B_1, B_2, B_3, ..., being Bernoulli's numbers), and taking $r=2m$, we get

$$\int\frac{\sin x}{x}dx=c+\frac{x}{(2m).2!}\left\{\left[\frac{\sin x}{x}+1\right]+2\left\{\frac{\sin\left[\frac{x}{2m}\right]}{\frac{x}{2m}}+\frac{\sin\left[\frac{2x}{2m}\right]}{\frac{2x}{2m}}+\frac{\sin\left[\frac{3x}{2m}\right]}{\frac{3x}{2m}}\right.\right.$$

$$\left.\left.+\cdots\cdots+\frac{\sin\left[\frac{2m-1}{2m}x\right]}{\frac{(2m-1)x}{2m}}\right\}-\frac{x^2}{6.(2m)^2.2!}\left[\frac{x\cos x-\sin x}{x^2}\right]\right.$$

$$+\frac{x^2}{30.(2m)^4.4!}\left[\left(\frac{6-x^2}{x^3}\right)\cos x+\left(\frac{6-3x^2}{x^4}\right)\sin x\right]\cdots\cdots(2).$$

Annals of Mathematics, Vol. V, No. 4, July, 1904.

But as m is an integer, $\int_0^{m\pi}\frac{\sin x}{x}dx$ develops by means of (2) into

$$\tfrac{1}{2}\pi+\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots\ldots-(-1)^m\frac{1}{(2m-1)}\right)\pm\pi\left(\frac{1}{6.2^2.m.2!}+\frac{(m\pi)^2-6}{30.2^4.m^3.4!}\right.$$

$$\left.+\frac{(m\pi)^4-20\,(m\pi)^2+120}{42.2^6.m^5.6!}+\ldots\ldots\right)\ldots\ldots(3),$$

according as m is odd or even.

For convenient use in numerical computation (3) may be put into the form

$$\int_0^{m\pi}\frac{\sin x}{x}dx=\tfrac{1}{2}\pi+\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots\ldots-(-1)^m\frac{1}{(2m-1)}\right)$$

$$\pm\left(\frac{c_1}{m}-\frac{c_3}{m^3}+\frac{c_5}{m^5}-\ldots\ldots\right)\ldots\ldots(4),$$

where $c_1=.0682995$, $c_3=.0019567$, $c_5=.0001948$, approximately.

By means of (4) the values of the definite integral corresponding to a few values of m are readily found to be as follows:

For $m=1$, $1.851936-$
$m=2$, $1.418158+$
$m=3$, $1.674760-$
$m=4$, $1.492164+$
$m=5$, $1.633963-$
$m=6$, $1.518036-$
.....

SOLUTIONS OF PROBLEMS.

ALGEBRA.

247. Proposed by PROFESSOR G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Find the sum, to n terms, of

$$1+\frac{n}{2}+\frac{n(n+2)}{2.4}+\frac{n(n+2)(n+4)}{2.4.6}+\ldots\ldots$$

I. Solution by the PROPOSER.

The series is the coefficient of x^{n-1} in $(1-x)^{-\frac{1}{2}n}(1-x)^{-1}$; *i. e.*, in $(1-x)^{-(\frac{1}{2}n+1)}$. Hence the required sum is

$$\frac{(n+2)(n+4)\ldots.(3n-2)}{2.4\ldots.(2n-2)}.$$

II. Solution by HENRY HEATON, Atlantic, Iowa.

The sum of the first two terms is $\frac{n+2}{2}$; of the first three, $\frac{(n+2)(n+4)}{2.4}$; of the first four, $\frac{(n+2)(n+4)(n+6)}{2.4.6}$; of the first r, $\frac{(n+2)(n+4)\ldots\ldots(n+2r-2)}{2.4\ldots\ldots(2r-2)}$. This may be rigorously demonstrated by induction. If $r=n$ the required sum is

$$\frac{(n+2)(n+4)\ \ldots\ldots\ (3n-2)}{2.4\ldots\ldots\ (2n-2)}.$$

Also solved by L. E. Newcomb, J. Scheffer, and G. B. M. Zerr.

248. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\frac{6435}{2}\cdot\frac{161280^2}{929569}\left[1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots\right]=\pi^{16}$.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

In books on higher algebra it is proved that

$$1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\ldots\ldots=\frac{(2^{2n}-1)B_{2n-1}}{2n!}\cdot\frac{\pi^{2n}}{2},$$

B_{2n-1} being* the nth of Bernoulli's numbers.

$$\therefore 1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots=\frac{(2^{16}-1)\,B_{15}}{16!}\cdot\frac{\pi^{16}}{2},$$

$$\therefore \pi^{16}=\frac{2\times 16!}{(2^{16}-1)\,B_{15}}\left(1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots\right).$$

Since $B_{15}=\frac{3617}{510}$, we find $\frac{2\times 16!\times 510}{(2^{16}-1)3617}$, which agrees with the coefficient given in the problem.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots\ldots=\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\ldots$$

$$\therefore \log\left(1-\frac{4\theta^2}{\pi^2}\right)+\log\left(1-\frac{4\theta^2}{3^2\pi^2}\right)+\log\left(1-\frac{4\theta^2}{5^2\pi^2}\right)+\ldots\ldots$$

$$=\log\left\{1-\left[\frac{\theta^2}{2!}-\frac{\theta^4}{4!}+\frac{\theta^6}{6!}-\frac{\theta^8}{8!}+\frac{\theta^{10}}{10!}-\frac{\theta^{12}}{12!}+\frac{\theta^{14}}{14!}-\frac{\theta^{16}}{16!}+\ldots\ldots\right]\right\}$$

$$=\log(1-A).$$

*Chrystal's *Algebra*, Vol. II, Chapter 30, §15.

$$\therefore -\frac{4\theta^2}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)-\frac{4^2\theta^4}{2}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\ldots\right)-\ldots\ldots$$

$$-\frac{4^8\theta^{16}}{8\pi^{16}}\left(1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}\right)=-A-\tfrac{1}{2}A^2-\ldots\ldots-\tfrac{1}{8}A^8-\ldots\ldots$$

Equating like coefficients of θ^{16} we get

$$\frac{8192}{\pi^{16}}\left(1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots\right)=\frac{929569}{6435\times 1260^2}.$$

$$\therefore \frac{6435}{2}\cdot\frac{161280^2}{929569}\left(1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots\right)=\pi^{16}.$$

Also solved by F. Anderegg, and G. W. Greenwood.

249. Proposed by J. J. KEYES, Fogg High School, Nashville, Tenn.

Solve $x+y+z=5$, $x^2+y^2=z^2$, $x^3+y^3+z^3=8$.

Solution by M. R. BECK, Cleveland, Ohio.

$x+y+z=5$, or $x+y=5-z$.....(1), $x^2+y^2=z^2$.....(2),
$x^3+y^3+z^3=8$, or $(x+y)(x^2-xy+y^2)=8-z^3$.....(3).

From (1) and (2), we have $xy=\dfrac{25-10z}{2}$.....(4).

Substituting (1), (2), and (4) in (3), and solving, $z=\frac{4}{3}\frac{7}{5}$. Substituting $z=\frac{4}{3}\frac{7}{5}$ in (1) and (2) and solving, $x=\frac{3}{2}\frac{9}{5}\mp\frac{7}{5}\sqrt{-34}$, and $y=\frac{3}{2}\frac{9}{5}\pm\frac{7}{5}\sqrt{-34}$.

Also solved by Henry Heaton, A. H. Holmes, L. E. Newcomb, J. Scheffer, and G. B. M. Zerr.

AVERAGE AND PROBABILITY.

172. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

What is the average length of all straight lines that can be drawn within a given triangle?

II. Solution by HENRY HEATON, Atlantic, Iowa.

Let ABC be the triangle. Let x=length of one of the straight lines, and θ the angle it would make with the side AB if produced to meet it, θ being taken $>$ than the angle A and $<$ than $\pi-B$. At the distance $x\sin\theta$ from AB draw ED parallel to AB cutting AC in E and BC in D. Then the number of lines of length x making the angle θ with AB is equal to the number of points in the triangle DEC whose area is $\dfrac{\triangle\,(b\sin A-x\sin\theta)^2}{b^2\sin^2 A}$. Similarly, the number of lines of length x making the angle θ with AC is equal to the number of points in the triangle whose area is $\dfrac{\triangle\,(c\sin A-x\sin\theta)^2}{c^2\sin^2 A}$, and the number making same angle with the side BC is equal to the number of points in the triangle whose area is

$\frac{\triangle\ (c\sin B - x\sin\theta)^2}{c^2\sin^2 B}$. Then if all lines are equally distributed about the starting point the required average is

$$M_1 = \frac{\int_A^{\pi-B}\int_0^{\frac{b\sin A}{\sin\theta}} \frac{(b\sin A - x\sin\theta)^2}{b^2\sin^2 A}\,d\theta\,x\,dx + \int_A^{\pi-C}\int_0^{\frac{c\sin A}{\sin\theta}} \frac{(c\sin A - x\sin\theta)^2}{c^2\sin^2 A}\,d\theta\,x\,dx}{\int_A^{\pi-B}\int_0^{\frac{b\sin A}{\sin\theta}} \frac{(b\sin A - x\sin\theta)^2}{b^2\sin^2 A}\,d\theta\,dx + \int_A^{\pi-C}\int_0^{\frac{c\sin A}{\sin\theta}} \frac{(c\sin A - x\sin\theta)^2}{c^2\sin^2 A}\,d\theta\,dx}$$

$$+ \int_B^{\pi-C}\int_0^{\frac{c\sin B}{\sin\theta}} \frac{(c\sin B - x\sin\theta)^2}{c^2\sin^2 B}\,d\theta\,x\,dx$$

$$+ \int_B^{\pi-C}\int_0^{\frac{c\sin B}{\sin\theta}} \frac{(c\sin B - x\sin\theta)^2}{c^2\sin^2 B}\,d\theta\,dx$$

$$= \frac{\int_A^{\pi-B} \frac{b^2\sin^2 A}{\sin^2\theta}\,d\theta + \int_A^{\pi-C} \frac{c^2\sin^2 A}{\sin^2\theta}\,d\theta + \int_B^{\pi-C} \frac{c^2\sin^2 B}{\sin^2\theta}\,d\theta}{4\int_A^{\pi-B} \frac{b\sin A}{\sin\theta}\,d\theta + 4\int_A^{\pi-C} \frac{c\sin A}{\sin\theta}\,d\theta + 4\int_B^{\pi-C} \frac{c\sin B}{\sin\theta}\,d\theta}$$

$$= \tfrac{3}{4}abc \div ab\log\left(\frac{s}{s-c}\right) + ac\log\left(\frac{s}{s-b}\right) + bc\log\left(\frac{s}{s-a}\right).$$

If the lines are so distributed as to join every possible pair of points the required average is

$$M_2 = \frac{\int_A^{\pi-B}\int_0^{\frac{b\sin A}{\sin\theta}} \frac{(b\sin A - x\sin\theta)^2}{b^2\sin^2\theta}\,d\theta\,x^2\,dx + \int_A^{\pi-C}\int_0^{\frac{c\sin A}{\sin\theta}} \frac{(c\sin A - x\sin\theta)^2}{c^2\sin^2\theta}\,d\theta\,x^2\,dx}{\int_A^{\pi-B}\int_0^{\frac{b\sin A}{\sin\theta}} \frac{(b\sin A - x\sin\theta)^2}{b^2\sin^2 A}\,d\theta\,x\,dx + \int_A^{\pi-C}\int_0^{\frac{c\sin A}{\sin\theta}} \frac{(c\sin A - x\sin\theta)^2}{c^2\sin^2 A}\,d\theta\,x\,dx}$$

$$+ \int_B^{\pi-C}\int_0^{\frac{c\sin B}{\sin\theta}} \frac{(c\sin B - x\sin\theta)^2}{c^2\sin^2 B}\,d\theta\,x^2\,dx$$

$$+ \int_B^{\pi-C}\int_0^{\frac{c\sin B}{\sin\theta}} \frac{(c\sin B - x\sin\theta)^2}{c^2\sin^2 B}\,d\theta\,x\,dx$$

$$= \tfrac{2}{5}\,\frac{\left[\int_A^{\pi-B} \frac{b^3\sin^3 A}{\sin^3\theta}\,d\theta + \int_A^{\pi-C} \frac{c^3\sin^3 A}{\sin^3\theta}\,d\theta + \int_B^{\pi-C} \frac{c^3\sin^3 B}{\sin^3\theta}\,d\theta\right]}{\int_A^{\pi-B} \frac{b^2\sin^2 A}{\sin^2\theta}\,d\theta + \int_A^{\pi-C} \frac{c^2\sin^2 A}{\sin^2\theta}\,d\theta + \int_B^{\pi-C} \frac{c^2\sin^2 B}{\sin^2\theta}\,d\theta}$$

$$= \tfrac{1}{15}\left(a+b+c + (a+b)\frac{(a-b)^2}{c^2} + (a+c)\frac{(a-c)^2}{b^2} + (b+c)\frac{(b-c)^2}{a^2}\right)$$

$$+ \tfrac{4}{15}\triangle^2\left[\frac{1}{c^3}\log\left(\frac{s}{s-c}\right) + \frac{1}{b^3}\log\left(\frac{s}{s-b}\right) + \frac{1}{a^3}\log\left(\frac{s}{s-a}\right)\right].$$

In the above free use is made of the well known formulas

$$\sin A = \frac{2\triangle}{bc}, \quad \cos A = \frac{b^2+c^2-a^2}{bc}, \quad \tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \text{ etc.}$$

*If $c=b=a$, $M_1 = \dfrac{a}{4\log 3} = .2276a$, and $M_2 = \tfrac{1}{5} + \tfrac{4}{5}\dfrac{\triangle^2}{a^3}\log 3 = \dfrac{a}{5}(1+\tfrac{3}{4}\log 3)$ $= .3638a$.

CALCULUS.

Note on Problem 207.—The proposer's value of $\frac{1}{4}\pi^2$ for the definite integral $\int_0^1 \frac{Kd_\kappa}{1+\kappa}$ is wrong and should be $\frac{1}{8}\pi^2$. Mr. Corey's solution (p. 230) is correct except in the last line where the evaluation of $u-v = 2\int_0^1 \frac{\log(1+z)}{z}dz - \int_0^1 \frac{\log(1+z^2)}{z}$ should read $2\frac{\pi^2}{12} - \frac{1}{2}\frac{\pi^2}{12} = \frac{\pi^2}{8}$. That this is true is easily shown by the following evaluation, due to Dr. Zerr, who has contributed two correct solutions of the problem.

$$u-v = 2\int_0^1 (1-\tfrac{1}{2}z+\tfrac{1}{3}z^2-\tfrac{1}{4}z^3+\ldots)dz - \int_0^1 (z-\tfrac{1}{2}z^3+\tfrac{1}{3}z^5-\tfrac{1}{4}z^7+\ldots)dz$$

$$= 2(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots) - \tfrac{1}{2}(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots)$$

$$= -\tfrac{3}{2}(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots - \frac{2}{1^2}-\frac{2}{3^2}-\frac{2}{5^2}-\ldots)\dagger = -\tfrac{3}{2}(\frac{\pi^2}{6}-\frac{\pi^2}{4}) = \frac{\pi^2}{8}.$$

G.

210. Proposed by EDWIN L. RICH, Schenectady, New York.

Determine a polynomial, $f(x)$, entirely in x and of the seventh degree, so that $f(x)+1$ is divisible by $(x-1)^4$ and $f(x)-1$ by $(x+1)^4$. [Goursat-Hedrick, *A Course in Mathematical Analysis*, p. 32, Ex. 3.]

*See Problem 35, p. 391, Williamson's *Integral Calculus*.

†See for example, Locke's *Treatise on Higher Trigonometry*, p. 99 and Ex. 1, p. 100.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

$$f(x)\equiv(x-1)^4(ax^3+bx^2+cx+d)-1\equiv(x+1)^4(a'x^3+b'x^2+c'x+d')+1.$$

Putting $-x$ for x and transposing we have

$$(x-1)^4(a'x^3-b'x^2+c'x-d')-1\equiv(x+1)^4(ax^3-bx^2+cx-d)+1.$$

These identities are consistent if $a=a'$, $b=-b'$, $c=c'$, $d=-d'$.

By equating coefficients of the same powers of x in either identity, and solving the resulting equations we get $a=\frac{5}{16}$, $b=\frac{5}{4}$, $c=\frac{9}{16}$, $d=1$.

$\therefore f(x)\equiv\frac{5}{16}x^7-\frac{9}{16}x^5+\frac{5}{16}x^3-\frac{5}{16}x.$

Also solved by R. D. Carmichael, A. H. Holmes, Henry Heaton, J. Scheffer, G. B. M. Zerr, and the Proposer.

211. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

*If $x=v^{1/(v-1)}$, what is the $f(x)$ such that $v=f(x)$?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $v=u+1$. Then $x^u=u+1$. Let $x=e^c$. $\therefore e^{cu}=u+1$. Let $cu=y$. $\therefore e^y=y/c+1$. $\therefore y=-c+ce^y$. If $y=a+b\varphi(y)$ we have by Lagrange's Theorem

$$y=a+b\varphi(a)+\frac{b^2}{2!}\cdot\frac{d}{da}[\varphi(a)]^2+\ldots..+\frac{b^n}{n!}\cdot\frac{d^{n-1}}{da^{n-1}}[\varphi(a)]^n+\ldots.. \text{ etc.}$$

In this problem $\varphi(a)=e^a$.

$$\therefore y=-c+ce^{-c}+c^2e^{-2c}+\frac{c^3}{2!}e^{-3c}+\frac{c^4}{3!}e^{-4c}+\ldots..+\text{ etc.}$$

$$\therefore u+1=v=f(x)=e^{-c}+ce^{-2c}+\frac{c^2}{2!}e^{-3c}+\frac{c^3}{3!}e^{-4c}+\ldots.. \qquad c=\log_e x,\ e^{-c}=1/x.$$

$$\therefore v=f(x)=1/x+\frac{\log_e x}{x^2}+\frac{(\log_e x)^2}{2!x^3}+\frac{(\log_e x)^3}{3!x^4}+\ldots..+\frac{(\log_e x)^n}{n!x^{n+1}}+\ldots.. \text{ etc.}$$

$$\therefore v=f(x)=x^{(1-x)/x}.$$

GEOMETRY.

267. Proposed by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Prove that every orthogonal system of circles in a plane is an isothermal system.

*This problem should admit of interesting generalizations, say for $x=v^{f(v)}$ for certain classes of functions f. G.

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Let any given orthogonal system of circles be inverted with respect to a circle whose center is an intersection of two circles, one from each family, and neither of them a real circle. The result is a new orthogonal system containing two straight lines derived from the two circles, and each line is the locus of centers of the 'opposite' family of circles. Using these lines as axes of coördinates, the two circle families are $(x-a)^2+y^2=c^2$, $x^2+(y-b)^2=d^2$, in which, because the circles are orthogonal, $a^2+b^2=c^2+d^2$. Writing $c^2=a^2-k^2$ in the last equation gives $d^2=b^2+k^2$, and the circle families become $x^2+y^2-2ax+k^2=0$, $x^2+y^2-2by-k^2=0$.

The constants are now independent, but since any circle of one family is orthogonal to all of the other family it follows that a and b are the respective parameters. If now a and b are replaced by $k\cot h2v$ and $-k\cot 2u$, respectively, the system may be written $u+vi=\tan^{-1}\frac{x+yi}{k}$, which shows that it is isothermal. Thus the given system is also isothermal, since it may be obtained from this one by inversion. When k is 0 or ∞ the corresponding result is

$$u+vi=\frac{1}{x+yi}, \text{ or } u+vi=x+yi.$$

274. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If a straight line AB is placed between two intersecting straight lines MN and PQ and is made to revolve through all possible positions having A always in MN and B always in PQ, what is the locus of any point L in AB or AB produced?

I. Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

We can choose coördinate axes so that the equations to the given lines are $y=rx$, $z=a$; $y=-rx$, $z=-a$. Let the coördinates of A, B, L be, respectively, (h, rh, a), $(k, -rk, -a)$, (x, y, z). Then

$$\frac{x-h}{x-k}=\frac{y-rh}{y+rk}=\frac{z-a}{z+a}=\frac{AL}{BL}=m, \text{ say.}$$

$\therefore h-mk=x(1-m)$.....(1), $r(h+mk)=y(1-m)$.....(2), $z(1-m)=a(1+m)$.....(3).

Hence the locus lies in a plane parallel to $z=0$, or to the given lines as is otherwise evident. Also $AB^2=l^2=(h-k)^2+r^2(h+k)^2+4a^2$.....(4). Eliminating h, k between (1), (2), (4) we have an ellipse for the required locus, its equation being

$$(1-m)^2\{[y(m-1)+rx(m+1)]^2+[y(m+1)+rx(m-1)]^2r^2\}=4m^2r^2(l^2-4a^2).$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the intersection of MN, PQ. $OA=a$, $OB=b$. Draw LD parallel to OB, and let D be in MN. Let $OD=u$, $DL=v$, $AB=c$, $AL=d$, $\angle AOB=\beta$. Then $a^2+b^2-2ab\cos\beta=c^2$; $c : d=a : a\pm u$; $c : d=b : v$.

$$\text{Hence } a=\pm\frac{cu}{d-c},\ b=\frac{cv}{d},\ \text{and}\ \frac{c^2u^2}{(d-c)^2}+\frac{c^2v^2}{d^2}\mp\frac{2c^2uv\cos\beta}{d(d-c)}=c^2.$$

$$\therefore \frac{u^2}{(d-c)^2}+\frac{v^2}{d^2}\mp\frac{2uv\cos\beta}{d(d-c)}=1.$$

$\therefore$ The locus is an ellipse.

III. Solution by A. H. HOLMES, Brunswick, Maine.

Suppose the straight lines MN and PQ intersect each other at right angles at O, and AB placed between them: A on MN and B on PQ, and L a point in AB. Draw LO. Put $AL=b$, $BL=a$, and $LO=r$, and $LAO=\phi$, $AOL=\theta$. Then $b\sin\phi=r\sin\theta$, and $a\cos\phi=r\cos\theta$.

$$\therefore \sin^2\phi=\frac{r^2}{b^2}\sin^2\theta,\ \text{and}\ \cos^2\phi=\frac{r^2}{a^2}\cos^2\theta.\quad \therefore r=\frac{ab}{\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}}.$$

Therefore the locus of point L is an ellipse whose semi-major axis is BL and whose semi-minor axis is AL. When MN and PQ intersect obliquely at angle ψ the semi-minor axis would be $\dfrac{ab\sin\psi}{\sqrt{(a^2-b^2\cos^2\psi)}}$.

Also solved by R. D. Carmichael, and J. Scheffer.

275. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

An hyperbola is drawn touching the axes of an ellipse, and the asymptotes of the hyperbola touch the ellipse. Prove that the center of the hyperbola lies on one of the equal conjugate diameters of the ellipse.

Solution by the PROPOSER.

Let $(x',\ y')$ be the intersection of the tangents to the ellipse $a^2y^2+b^2x^2-a^2b^2=0$ ..... (1); then these tangents being the asymptotes of the hyperbola, $(x',\ y')$ is the center of the hyperbola. The equation to the tangents to (1) from $(x',\ y')$ is

$$(a^2y^2+b^2x^2-a^2b^2)(a^2y'^2+b^2x'^2-a^2b^2)=(a^2y'y+b^2x'x-a^2b^2)^2 \text{.....(2),}$$

$$\text{or, } (y'^2-b^2)x^2+(x'^2-a^2)y^2-2x'y'xy+2b^2x'x+2a^2y'y-(a^2y'^2+b^2x'^2)=0\text{.....(3).}$$

Now, the equation to the asymptotes of a conic differs from the equation to the conic by a constant only; then adding c to the left member of (3) we have the equation to the hyperbola.

If now $y=0$ in this equation to the hyperbola, we have

$$(y'^2-b^2)x^2+2x'x-(a^2y'^2+b^2x'^2)+c=0 \ldots\ldots (4),$$

and the condition that (4) has equal roots, or that the hyperbola touches the X-axis is given by $y'^2(a^2b^2-a^2y'^2-b^2x'^2)=c(b^2-y'^2)$ .....(5); and, in a similar way that the curve touches the Y-axis, $x'^2(a^2b^2-a^2y'^2-b^2x'^2)=c(a^2-x'^2)$ .....(6). (5)÷(6) gives after reduction, $a^2y'^2-b^2x'^2=0$.....(7), showing that (x', y') is on an equi-conjugate axis of the ellipse.

Also solved by G. W. Greenwood, A. H. Holmes, W. W. Landis, J. Scheffer, and G. B. M. Zerr.

GROUP THEORY.

12. Proposed by GEORGE H. HALLETT, Ph. D., Assistant Professor of Mathematics, The University of Pennsylvania.

Given $U_1=a'$, $V_1=\beta'$, and the recursion formulae $U_y=a'V_{y-1}+a''U_{y-1}$, $V_y=\beta'V_{y-1}+\beta''U_{y-1}$. Find expressions for U_y, V_y in terms of the coefficients a', a'', β', β''.

Solution by PROFESSOR JAMES BYRNIE SHAW, The James Milliken University, Decatur, Ill.

By eliminating V we find that U_n is the coefficient of x^n in the expansion of

$$\frac{a'x}{1-(a''+\beta')x+(a''\beta'-a'\beta'')x^2}.$$

Likewise we find that V_n is the coefficient of x^n in the expansion of

$$\frac{1-a''x}{1-(a''+\beta')x+(a''\beta'-a'\beta'')x^2}.$$

We may state the result as follows: Let $\cos\theta=\frac{1}{2}.\frac{a''+\beta'}{T}$ where $T^2=\begin{vmatrix} a'' & a' \\ \beta'' & \beta' \end{vmatrix}$. Then $V_n=a'.T^{n-1}.\frac{\sin n\theta}{\sin\theta}$, and $V_n=T^n.\frac{\sin(n+1)\theta}{\sin\theta}-a''T^{n-1}\frac{\sin n\theta}{\sin\theta}$. These latter forms are easily verified by mathematical induction. The well-known formulae for $\frac{\sin n\theta}{\sin\theta}$ give U_n and V_n in terms of the coefficients directly, and free from irrationalities.

13. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

The order of the linear homogeneous group in n letters is $(p^n-1)(p^n-p)$ .....(p^n-p^{n-1}). Two proofs are given in Burnside's *Finite Groups*. Give other proofs.

Solution by the PROPOSER.

The linear homogeneous group is known to be equivalent to the group of isomorphisms of the abelian group $H_{p^n}=[P_1, P_2 \ldots\ldots P_n]$ of type [1 1 1.....],

order p^n. Let $h_{i_1}\ h_{i_2}$ ..... represent the subgroups of H, of order p, and $J_{ik}=\begin{pmatrix} P_1 & P_2 & \cdots\cdots & P_n \\ h_{i_1} & h_{i_2} & \cdots\cdots & h_{i_n} \end{pmatrix}$ the isomorphism of H gotten by replacing P_j by any operation (order p) in h_{ij} ($j=1\ 2$ .....m), say the new generators from the kth set of all of the possible sets which might be chosen from h_{i_1}, h_{i_2} ..... h_{i_n}. The number of values of k is obviously equal to $\Phi(p)^n=(p-1)^n$. To determine the number of choices *of this set* of subgroups (number of values of i) suppose that a of a set of n generators have been selected. The remaining $n-a$ operations must be selected outside the subgroup H_{p^a} generated by the first a, and thus there remain

$$\frac{p^n-1}{p-1}-\frac{p^a-1}{p-1}=\frac{p^a(p^{n-a}-1)}{p-1}$$

subgroups h_{ij} from which to select the remaining $n-a$. Thus the product of the number of values of k and the number of values of i is

$$h=(p-1)^n\prod_{a=0}^{n-1}\frac{p^a(p^{n-a}-1)}{p-1}=(p^n-1)(p^n-p)(p^n-p^2)\cdots\cdots(p^n-p^{n-1})$$

which is the number of choices of new generators of H, or the order of its automorph.

MECHANICS.

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186. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

A point P keeps at uniform distance from and moves with uniform angular velocity around a point Q which is in harmonic motion, making one revolution while Q swings to and fro. If P is in the line of the path of Q and on the same side of the center of that path with Q when Q is at the extremity of the path, what is the locus of P?

Solution by the PROPOSER.

Take the origin at the center of the path of Q, and let a=half the length of that path. Let $PQ=b$, and let θ=the angle of PQ with the path of Q at any time. Then, it is easily shown that $x=(a+b)\cos\theta$, $y=b\sin\theta$, the equations of an ellipse whose axes are $a+b$ and b.

Also solved by G. W. Greenwood, and G. B. M. Zerr.

PROBLEMS FOR SOLUTION.

ALGEBRA.

253. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Prove that $x^5+ax+b=0$ is solvable by radicals if $b=ma$, m being the negative of half the sum of any two roots of the original equation. Exhibit the solution.

254. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series $\dfrac{n^2}{(16n^2-1)^2}$ beginning with $n=1$.

255. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Let f be the binary cubic $a_0x_1^3+3a_1x_1^2x_2+3a_2x_1x_1^2+a_3x_2^3$, $\Delta=(f,f)_2$ the covariant, the second transvectant of f over itself, and $R=2[4(a_0a_2-a_1^2)\times(a_1a_3-a_2^2)-(a_0a_3-a_1a_2)^2]=(\Delta,\Delta)_2$=the second transvectant of Δ over itself. Then if $\Delta_{\kappa\lambda}$ is the Δ covariant for the cubic pencil $\kappa f+\lambda Q$, Q being the first transvectant of f over Δ we have $\Delta_{\kappa\lambda}=(\kappa^2-\frac{1}{2}\lambda^2R)\Delta$.

CALCULUS.

900. Proposed by PROFESSOR B. F. FINKEL, A. M., 4038 Locust Street, Philadelphia, Pa.

Prove that, if the differential equation $cydx-(y+a+bx)dy-nx(xdy-ydx)=0$, be transformed into an equation between u and x by the substitution $u(y+a+bx+nx^2)=y(c+nx)$, then the variables are separable; and reduce the equation to the form $dv/\phi(v)=dx/\phi(x)$ by the further substitution $v=au+\beta$, a and β being suitably determined. *Euler.* [Forsyth's *Differential Equations*, p. 48, Ex. 4.]

DIOPHANTINE ANALYSIS.

132. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Disregarding the order of λ, μ, ν, how many sets of solutions has the congruence $\lambda+\mu+\nu\equiv 0 \pmod{p-1}$ (p prime)?

GEOMETRY.

280. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

On any diameter of a given ellipse is taken a point such that the tangents from it intercept on the tangent at one end of the diameter a length equal to the diameter; the ellipse being $a^2y^2+b^2x^2-a^2b^2=0$. Prove that the locus of the point is $\left(\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}\right)^2=\left(\dfrac{a^2+b^2}{a^2-b^2}\right)^2\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)$.

281. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

In the proposition in solid geometry "If a line is perpendicular to each of two intersecting lines it is perpendicular to the plane of the lines," it is assumed that two intersecting lines have a common perpendicular. Prove it.

282. Proposed by REV. ALAN S. HAWKESWORTH, Allegheny, Pa.

The pedal lines of any two points on the circumcircle of a triangle concur in an angle equal to that subtended by the said points.

283. Proposed by REV. ALAN S. HAWKESWORTH, Allegheny, Pa.

The right angled intersection of the pedal lines of any diameter of the circumcircle lies on the "nine points circle" of the inscribed triangle.

MECHANICS.

990. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

A curve in a vertical plane has a horizontal tangent along which a heavy particle moves without friction with uniform velocity v until it reaches T, the point of tangency, when by its own momentum alone it ascends the curve against the force of gravity. Find the equation of the curve such that the particle, after ascending for a time along the curve, will leave it and fall freely till it strikes at T.

NOTES AND NEWS.

Mr. E. B. Smith, instructor in mathematics at Purdue University, has resigned.

Mr. W. M. Persons, of the University of Wisconsin, has been appointed assistant professor of mathematics at Dartmouth College.

Professor J. E. Bonebright, of the Colorado Agricultural College, has been appointed professor of mathematics at Ottawa University, Ottawa, Kansas.

The following officers were elected at the Chicago meeting of the Central Association of Science and Mathematics Teachers: President, O. W. Caldwell, Charleston, Ill.; Treasurer, C. W. D. Parsons, Evanston, Ill.

Dr. S. T. Tamura, B. Sc., M. A. (Iowa), Ph. D. (Columbia), a native of Japan, has been appointed mathematician in the department of terrestrial magnetism of the Carnegie Institution, with which he has been connected as assistant for the past two years.

The Mathematical Section of the California Teachers Association elected the following officers at its recent meeting: President, Professor G. A. Miller; Vice President, Professor W. H. Baker, San Jose Normal School; Secretary, Mr. J. Fred Smith, Campbell High School.

Professor Alexander Ziwet, Chairman and Vice President of Section A, American Association for the Advancement of Science, took for the subject of his New Orleans vice presidential address "The Relation of Mechanics to Physics." His address is published in full in *Science* of January 12.

The Colorado Mathematics Society, having for its main object the improvement of the teaching of mathematics, was organized in Denver, Colorado, on December 2, 1905. Professor DeLong, of the University of Colorado, was elected president, and Mr. Smith, of the North Denver High School, secretary.

The Proceedings of the International Congress of Arts and Sciences, which held its meeting at St. Louis in September, 1904, are being published by Houghton, Mifflin Company. These will comprise eight volumes, the first of which will contain papers on philosophy and mathematics. The papers will be published as presented, except that those read in foreign languages will be translated into English.

A NATIONAL SOCIETY, PRO AND CON.

The report of the committee on "American Society of Teachers of Mathematics and the Natural Sciences" is published in the January number of *School Science and Mathematics*. This committee has drafted a constitution which it has submitted to the leading local mathematics and science teachers associations in the country for their approval. The objects of the American Society as set forth in Article II of the Constitution are, (1) "To improve the teaching of mathematics and science." (2) "To promote the interests of teachers of mathematics and the natural sciences." (3) "To foster the organization of such teachers in groups of various scopes." (4) "To support a journal or journals devoted to the aims of the society." It is proposed in the Constitution to hold one annual meeting in conjunction with the N. E. A.

A significant response to the circulation of the committee's report was the resolution of the Central Association at Chicago December 2, that "the proposed constitution does not meet the ideals of the Central Association of Science and Mathematics Teachers." This association passed resolutions favoring the organization of three coöperating general societies, eastern, western, and central, each holding its own general meetings within its own territory, the three uniting as a national body in one annual meeting.

ASSOCIATION OF TEACHERS OF MATHEMATICS IN INDIANA.

The Mathematical Section of the Indiana State Teachers' Association held its annual meeting as a section of the Association at Indianapolis, on December 27, about 250 teachers of mathematics being present. The following papers were read: (1) The High School's Portion of Higher Mathematics, by Professor David A. Rothrock, Indiana University; (2) Instructing vs. Teaching,

by Professor John C. Stone, Indiana State Normal; (3) In What Grades Should the Study of Algebra Begin? by Superintendent George L. Roberts, Muncie; (4) How Can High School Mathematics Better Prepare for Study of Science? by Leonard Young, Evansville High School. By a vote of the Section a committee of five, consisting of Professor D. A. Rothrock, Indiana University, Bloomington, Professor T. G. Alford, Purdue University, Lafayette, Professor W. P. Morgan, State Normal, Terre Haute, Principal D. R. Ellabarger, Richmond High School, and Mr. G. H. Mingle, Anderson High School, was appointed to formulate plans for an "Association of Teachers of Mathematics in Indiana." The committee was empowered to call a meeting and arrange a program for preliminary organization. The first meeting of the proposed Association will be held at Indianapolis, March 30, in connection with the meeting of the Southern Indiana Teachers' Association.

THE AMERICAN MATHEMATICAL SOCIETY.

The American Mathematical Society held its twelfth annual meeting at Columbia University on December 28-29, simultaneously with the meetings of the American Physical Society, and the Astronomical and Astrophysical Society of America. Large attendance and extensive programs characterized the threefold gathering. A joint meeting of the Mathematical and Physical societies was held on Friday afternoon for the purpose of hearing Professor V. F. Bjerknes, of the University of Stockholm, who spoke on "Experimental Demonstration of Hydro-dynamic Action at a Distance."

The American Mathematical Society elected the following officers and members of the Council: Vice Presidents, Charlotte A. Scott and Irving Stringham; Secretary, F. N. Cole; Treasurer, W. S. Dennett; Librarian, D. E. Smith; Committee of Publication, F. N. Cole, Alexander Ziwet, D. E. Smith; Members of the Council, to serve until December, 1908, C. L. Bouton, L. E. Dickson, Edward Kasner, E. J. Townsend.

The following have been elected members of the Society: Mr. R. L. Börger, University of Missouri; Professor W. B. Cairns, Ursinus College; Mr. A. J. Champreux, University of California; Dr. Emily Coddington, New York, N. Y.; Dr. F. J. Dohmen, University of Texas; Dr. O. E. Glenn, Drury College; Mr. E. S .Haynes, University of Missouri; Professor J. H. Jeans, Princeton University; Mr. A. R. Maxson, Columbia University; Professor J. F. Travis, Georgia School of Technology; Professor Vito Volterra, University of Rome; Miss Mary E. G. Waddell, Orono, Canada.

ERRATA.

On page 230, line 6, for $\frac{1}{4}\pi^2$ read $\frac{1}{8}\pi^2$.

On page 230, line 16, for $\frac{2\pi^2}{6} - \frac{\pi^2}{2.6} = \frac{\pi^2}{4}$, read $2.\frac{\pi^2}{12} - \frac{\pi^2}{2.12} = \frac{\pi^2}{8}$.

2

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THE GROUPS WHICH CONTAIN LESS THAN TWENTY OPERATORS OF ORDER THREE.

By G. A. MILLER.

All the operators of a given order which are contained in a group (G) generate an invariant subgroup if they do not generate the entire group. As the existence of invariant subgroups in G is generally very useful in determining other properties, it is of interest to know what groups contain a given number of operators of a certain order and are generated by these operators. Very little has been done along this line. One of the most interesting problems in this connection is the determination of all the groups which contain a small number of operators of order 2 and are generated by these operators. This problem has been solved in case the number of the operators of order 2 does not exceed 15.

The present note is devoted to a study of a few of the groups which seem almost as interesting as those just mentioned. It will be observed that very few theorems of group theory are employed in these considerations. According to a well known theorem due to Frobenius, the number of subgroups of order p in any group is of the form $1+kp$. Hence the groups which come under the present heading contain 2, 8, or 14 operators of order 3. In the first case the operators of order 3 generate an invariant subgroup of this order. The other two cases lead to much more interesting results.

§ 1. *Groups which contain just eight operators of order three and are generated by these operators.*